

# Convection in Drying and Freezing Ground

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## Abstract

In this paper we analyse the drying of a soil composed of particles, water and solute impurities, and study the occurrence of convective instabilities during evaporation. We find that the main driving force for instability is the formation of a concentration gradient at the soil surface due to the evaporation of water. A similar phenomenon may occur during the thawing of frozen ground in Arctic regions.

## 1 Introduction

The formation of a surface seal in soil, the hardening of the seal to form a crust and cracking of this crust, has been observed to occur under the combined influence of rain and dry weather [1, 2]. The sealing leads to a local increase in the bulk density and a decrease in porosity along with a decrease in the hydraulic conductivity [3]. After drying these surface seals transform into surface crusts. The sealing and crusting of soil surfaces increases the runoff in the soil surfaces. In fact, initially the soil loss reduces because soil strength increases during crusting. However, crusting eventually leads to the formation of cracks which increases runoff in the soil surfaces [4]. The detailed effect of crusting on soil loss depends on various factors like the distribution of crusted and non-crusted regions and the shrinkage cracks within the crusted layer [5]. Cracking depends critically on the rate of drying and so it has been observed to occur within a few days of a rainstorm in a dry season [6].

The effects of sealing and crusting on soil loss are most visible in agricultural lands. This is because the sealing and crusting is most likely to occur in the soil which is not vegetated in the agricultural environment. In Europe a quarter of its agricultural land exhibits some form of soil erosion risk [8]. In fact, a 20 mm rain has been observed to form surface seal and cause soil loss in certain hilly areas [9, 10]. In this case, the thickness of the surface seal was observed to be on the order of a few millimeters. However, the formation of a 7 cm thick surface seal during a wet season has also been observed [11]. Since the soil loss can effect the properties of soil and thus have a major impact on agricultural production, it is important to understand the physical processes at work in drying ground. In this paper we analyse convective instabilities that are observed to occur during the evaporation of a mixture of soil particles and water [22].

Evaporation of pure liquids can cause convection in those liquids [12, 13, 14, 15, 16, 17]. This is because the evaporation causes the surface temperature to drop and thus sets up an unstable density gradient which can induce a Rayleigh–Bernard instability. In addition, for most liquids surface tension is a function of temperature and so perturbations in temperature along the film surface also create perturbations in the surface tension. The liquid then flows from places of lower surface tension to places of higher surface tension. This flow is called the Marangoni effect. Thus, both the temperature gradient and surface tension variations contribute to the occurrence of convection in pure liquids. The combined phenomenon is called Bernard–Marangoni convection.

The formation of surface seals and crusts can be understood by analysing the drying of pools of muddy water. Evaporation in mixtures also leads to convection [18, 19, 20, 21]. In this case as the liquid dries up a concentration gradient is also set up along with a temperature gradient. The liquid moves under the combined influence of both these gradients. Furthermore, the Marangoni effect also occurs due to the dependence of the of surface tension on both the temperature and the concentration. Thus, a perturbation in either the temperature or the concentration can cause convection. Convection has been observed to occur in the course of evaporation of a binary mixture of soil particles and water [22]. In doing so a four weight percent mixture of bentonite and water was analysed at twenty degrees centigrade. It was found that convection occurred during the course of drying of this mixture and produced hexagonal patterns. However, it was also observed that the temperature gradient set up during the evaporation could not explain the occurrence of convection in this system. In this paper we will show that the concentration gradient is the main driving force behind such convection during the drying of ground.

We also study the thawing of frozen ground which occurs in the Arctic circle. Freezing and thawing of soil in the Arctic circle results in the formation of various surface patterns such as soil hummocks and stone circles [23, 24, 25, 26]. Various models have been proposed to explain the occurrence of these patterns. In one of these models the maximum density of water at four degree centigrade sets up the convection [27, 28, 29]. This model is based on the convection of water through soil pores which has not been observed for the soils under consideration. It has also been proposed that the sedimentation of soil during thawing sets up an unstable density profile which causes convection [30, 31, 32]. However, the measured soil density gradient is not large enough to initiate convection [33]. Another model proposes that the motion responsible for pattern formation occurs during the freezing of ground (differential frost heave) [34, 35]. This model yields a plausible mechanism for patterned ground but has yet to be experimentally confirmed, and does not explain certain observations such as the soil convection observed to occur in later summer [25].

The soil convection can potentially be explained by a phenomenon similar to the one that occurs in the formation of surface seals. The ice melts in late summer and at the same time evaporation takes place from the surface of the soil. This causes an unstable density profile to develop that is in principle large enough to initiate wholesale soil convection. This convection has been observed in fields and is thought to contribute to the patterns that form in Arctic region [25, 32]. It may be noted that similar patterns have been detected on Mars, suggesting that in the past the temperature on Mars may have been large enough for the ice to melt and soil convection to take place [36].

In this paper we will use tensor notations for performing the stability analysis [37]. We will also use a new approach based on Green's functions to perform stability analysis. As will be evident that the stability analysis of a ternary mixture is very complicated, and it would not be possible to perform it using the conventional methods that are usually employed for stability analysis.

## 2 Ternary Mixture

In this section we will first analyse the evaporation of a mixture of soil particles and water. As the liquid evaporates, both the temperature and the concentration change at the surface. This causes an instability to occur which drives convection. In order to analyse the occurrence of this stability, we first observe that the density of the liquid depends on both the temperature of this mixture and concentration of particles in water. However, in any real situation there will also be various solute impurities in the soil. Hence, we analyze a ternary mixture of water, soil particles and dissolved solutes. Thus, we can write,

$$\rho = \rho_0[1 - \alpha_c(C - C_0) + \alpha'_c(C' - C'_0) + \alpha_t(T - T_0)], \quad (1)$$

where  $C$  is the concentration of soil particles and  $C'$  is the concentration of solute impurities. The coefficients  $\alpha_c$ ,  $\alpha'_c$  and  $\alpha_t$  are the particle, solutal and thermal expansion coefficients, respectively, taken to be constants. We now let  $C \rightarrow -C\Delta C + C_0$ ,  $C' \rightarrow -C'\Delta C' + C'_0$  and  $T \rightarrow -T\Delta T + T_0$ . We also define  $B_c = \alpha_c\Delta C$ ,  $B'_c = \alpha'_c\Delta C'$  and  $B_t = \alpha_t\Delta T$  and so we get,  $\rho = \rho_0[1 + B_cC + B'_cC' + B_tT]$ . Our system is described by a divergenceless vector field, which represents the fluid velocity,  $\partial^i v_i = 0$ . In order to write the momentum balance equation, it is useful to define a substantive derivative as follows

$$Dv_i = \partial_t v_i + v^j \partial_j v_i, \quad (2)$$

where  $\partial_t = \partial/\partial t$ ,  $\partial_i = \partial/\partial x^i$ , and  $\partial^i \partial_i = \partial^2$ . In continuum mechanics, this describes describes the time rate of change of some physical quantity for a material element subjected to a space-and-time-dependent velocity field. Now we can write the momentum balance equation as

$$\rho_0 Dv_i = -\partial_i p + \partial^j \mu [1 - C/C_p]^{-2} (\partial_i v_j + \partial_j v_i) - \rho g \lambda_i. \quad (3)$$

Here  $\rho$  is the density,  $g$  is the acceleration due to gravity,  $\mu$  is a constant viscosity and  $C_p$  is a maximum close packing concentration (shrinkage limit) [38]. The factor  $[1 - C/C_p]^{-2}$  denotes the dependence of viscosity on concentration. Along with this equation there are also the following equations,

$$\begin{aligned} DT &= \kappa \partial^2 T, \\ DC &= \partial^i d_{11} [1 - C/C_p]^{-2} \partial_i C + d_{12} \partial^2 C', \\ DC' &= d_{21} \partial^2 C + d_{22} \partial^2 C', \end{aligned} \quad (4)$$

where  $\kappa$  is the thermal diffusivity of the fluid,  $d_{11}$  is the soil diffusion coefficient,  $d_{22}$  is the solute diffusion coefficient, and  $d_{21}, d_{12}$  are cross diffusion coefficients. We have neglected both the Dufour effect and Soret effect in the temperature equation as they are very weak relative to other effects. Here

the thermal diffusivity of the fluid, solute diffusion coefficient and the cross diffusion coefficients are constants. As we will be analysing this system far away from  $C_p$ , we will neglect the factor  $C/C_p$  and so in this limit even soil diffusion coefficient and the viscosity is a constant. We will also use the Boussinesq approximation and set density constant in all terms except  $\rho g \lambda_i$ . We can use the natural length scale  $l$ , which is the depth of the liquid, and thermal diffusivity  $k$  to non-dimensionalise these equations. To do so we will transform each quantity as  $x^i \rightarrow l x^i$ ,  $\partial_i \rightarrow l^{-1} \partial_i$ ,  $v_i \rightarrow k l^{-1} v_i$ ,  $t \rightarrow l^2 k^{-1} t$ , and  $p \rightarrow p \mu k l^{-2} + \rho_0 g (l - \lambda^i r_i)$ . The new velocity field again remains divergenceless,  $\partial^i v_i = 0$ . The non-dimensional form of the momentum conservation equation and the continuity equation for temperature and concentration for constant viscosity and diffusion coefficients becomes

$$\begin{aligned} Pr^{-1} D v_i &= -\partial_i p + \partial^2 v_i + R_c C \lambda_i + R'_c C' \lambda_i + R_t T \lambda_i, \\ DC &= Le_{11}^{-1} \partial^2 C + Le_{12}^{-1} \partial^2 C', \\ DC' &= Le_{21}^{-1} \partial^2 C' + Le_{22}^{-1} \partial^2 C', \\ DT &= \partial^2 T, \end{aligned} \tag{5}$$

where  $Pr = k \rho_0 \mu^{-1}$  is the Prandtl number,  $R_c = \mu^{-1} k^{-1} B_c \rho_0 g l^3$  and  $R'_c = \mu^{-1} k^{-1} B'_c \rho_0 g l^3$  are the concentration Rayleigh numbers,  $R_t = \mu^{-1} k^{-1} B_t \rho_0 g l^3$  is the temperature Rayleigh number,  $Le_{11} = k d_{11}^{-1}$ ,  $Le_{22} = k d_{22}^{-1}$  are the Lewis numbers and  $Le_{12} = k d_{12}^{-1}$ ,  $Le_{21} = k d_{21}^{-1}$  are new Lewis number corresponding to cross diffusivity. We want to analyze the steady state version of these equations. In order to do so, we impose the following boundary conditions,  $v_z = 0$ ,  $\partial_z v_x = 0$ ,  $\partial_z v_y = 0$ , and  $C = C' = T = 1$ ,  $C(1) = C'(1) = T(1) = 0$ ,  $p(1) = 0$ . We also take  $\lambda_i = (0, 0, 1)$  and  $r_i = (x, y, z)$ , thus our liquid is placed on a level plane. Under these boundary conditions we can obtain the steady state solutions. We will also add small perturbations to these steady state solutions. Now the steady state solutions with perturbations added to them are given by

$$\begin{aligned} v_i &= u_i \\ C &= 1 - \lambda^i r_i + \theta_c \\ C' &= 1 - \lambda^i r_i + \theta'_c \\ T &= 1 - \lambda^i r_i + \theta_t \\ p &= -\frac{(R_c + R'_c + R_t)}{2} |1 - \lambda^i r_i|^2 + \phi. \end{aligned} \tag{6}$$

Now we define  $D_b^a = a^{-1} \partial_t - b^{-1} \partial^2$ , and so we get

$$\begin{aligned} D_1^{Pr} u_i + \partial_i \phi - R_c \theta \lambda_i &= 0, \\ D_{Le11}^1 \theta_c - Le_{12}^{-1} \partial^2 \theta'_c - \lambda^i u_i &= 0, \\ D_{Le22}^1 \theta'_c - Le_{21}^{-1} \partial^2 \theta_c - \lambda^i u_i &= 0, \\ D_1^1 \theta_t - \lambda^i u_i &= 0. \end{aligned} \tag{7}$$

We define an operator  $E$  which takes the curl of any vector field  $a^i$  twice,

$$\begin{aligned} E a^i &= \epsilon^{imn} \epsilon_{nkl} \partial_m \partial^k a^l \\ &= (\delta_k^i \delta_l^m - \delta_l^i \delta_k^m) \partial_m \partial^k a^l \\ &= \partial^i \partial^p a_p - \partial^2 a^i. \end{aligned} \tag{8}$$

So, the gradient of a scalar field vanishes when its curl is taken twice,  $E\partial^i\phi = 0$ , because in the expression  $E\partial^i\phi = \epsilon^{imn}\epsilon_{nkl}\partial_m\partial^k\partial^l\phi$  there is a contraction between a pair of symmetric and antisymmetric tensor indices. We also have  $Eu_i = -\partial^2 u_i$ , because  $u_i$  is divergenceless, and  $\lambda^i(E\lambda_i\phi) = \lambda^i\partial_i\lambda^i\partial_j\phi - \partial^2\phi = \tilde{\partial}^2\phi$ . Now acting on the momentum balance by  $E$  and contracting it with  $\lambda_i$ , we get

$$D_1^{Pr}\partial^2\lambda^i u_i - R_c\tilde{\partial}^2\theta_c - R'_c\tilde{\partial}^2\theta'_c - R_t\tilde{\partial}^2\theta_t = 0. \quad (9)$$

Now we define the following differential operator

$$\begin{aligned} \mathcal{D}_0 &= D_{Le22}^1 D_{Le11}^1 - Le_{12}^{-1} Le_{21}^{-1} \partial^4, \\ \mathcal{D}_1 &= (D_{Le22}^1 - Le_{12}^{-1} \partial^2), \\ \mathcal{D}_2 &= (D_{Le11}^1 - Le_{21}^{-1} \partial^2), \end{aligned} \quad (10)$$

and the following Green's function's

$$\begin{aligned} \mathcal{D}_0 G_c(r, r') &= \delta^3(r - r'), \\ D_1^1 G_t(r, r') &= \delta^3(r - r'). \end{aligned} \quad (11)$$

Now we have

$$\begin{aligned} \mathcal{D}_0\theta_c &= \mathcal{D}_1\lambda^i u_i, \\ \mathcal{D}_0\theta'_c &= \mathcal{D}_2\lambda^i u_i, \\ D_1^1\theta_t &= \lambda^i u_i. \end{aligned} \quad (12)$$

So, we have

$$\begin{aligned} \theta_c(r) &= \int d^3r' G_c(r, r') \mathcal{D}_1\lambda^i u_i(r'), \\ \theta'_c(r) &= \int d^3r' G_c(r, r') \mathcal{D}_2\lambda^i u_i(r'), \\ \theta_t(r) &= \int d^3r' G_t(r, r') \lambda^i u_i(r'). \end{aligned} \quad (13)$$

because

$$\begin{aligned} \mathcal{D}_0\theta_c(r) &= \int dr' \mathcal{D}_0 G_c(r, r') \mathcal{D}_1\lambda^i u_i(r'), \\ &= \int d^3r' \delta^3(r - r') \mathcal{D}_1\lambda^i u_i(r') \\ &= \mathcal{D}_1\lambda^i u_i(r), \\ \mathcal{D}_0\theta'_c(r) &= \int dr' \mathcal{D}_0 G_c(r, r') \mathcal{D}_2\lambda^i u_i(r'), \\ &= \int d^3r' \delta^3(r - r') \mathcal{D}_2\lambda^i u_i(r') \\ &= \mathcal{D}_2\lambda^i u_i(r), \\ D_1^1\theta_t(r) &= \int dr' D_1^1 G_t(r, r') \lambda^i u_i(r') \\ &= \int d^3r' \delta^3(r - r') \lambda^i u_i(r') \\ &= \lambda^i u_i(r). \end{aligned} \quad (14)$$

Thus, we can write,

$$\begin{aligned}
D_1^{Pr} \partial^2 \lambda^i u_i(r) &= R_c \tilde{\partial}^2 \int d^3 r' G_c(r, r') \mathcal{D}_1 \lambda^i u_i(r') \\
&\quad + R'_c \tilde{\partial}^2 \int d^3 r' G_c(r, r') \mathcal{D}_2 \lambda^i u_i(r') \\
&\quad + R_t \tilde{\partial}^2 \int d^3 r' G_t(r, r') \lambda^i u_i(r').
\end{aligned} \tag{15}$$

Multiplying by  $\mathcal{D}_0$  and  $D_1^1$ , we get

$$\begin{aligned}
\mathcal{D}_0 D_1^1 D_1^{Pr} \partial^2 \lambda^i u_i &= R_c \tilde{\partial}^2 D_1^1 \mathcal{D}_1 \lambda^i u_i + R'_c \tilde{\partial}^2 D_1^1 \mathcal{D}_2 \lambda^i u_i \\
&\quad + R_t \tilde{\partial}^2 \mathcal{D}_0 \lambda^i u_i.
\end{aligned} \tag{16}$$

Now using the boundary conditions that the even derivatives of  $\lambda^i \partial_i$  vanishes on  $\lambda^i u_i$ , we write the solution as [37],

$$\lambda^i u_i = A \sin n\pi \exp[i(k_x x + k_y y) + \sigma t], \tag{17}$$

becomes and we also define

$$k^2 = k_x^2 + k_y^2, \quad k_n^2 = k^2 + n^2 \pi^2. \tag{18}$$

Thus the characteristic equation is

$$C(k_m, k, \sigma) = 0, \tag{19}$$

where

$$\begin{aligned}
C(k_m, k, \sigma) &= ((\sigma + k_n^2 L e_{22}^{-1})(\sigma + k_n^2 L e_{11}^{-1}) - L e_{12}^{-1} L e_{21}^{-1} k_n^4) \\
&\quad \times (\sigma + k_n^2)(\sigma P r^{-1} + k_n^2) k_n^2 \\
&\quad - R_c (\sigma + k_n^2) ((\sigma + k_n^2 L e_{22}^{-1}) - L e_{12}^{-1} k_n^2) k^2 \\
&\quad - R'_c (\sigma + k_n^2) ((\sigma + k_n^2 L e_{11}^{-1}) - L e_{12}^{-1} k_n^2) k^2 \\
&\quad - R_t ((\sigma + k_n^2 L e_{22}^{-1})(\sigma + k_n^2 L e_{11}^{-1}) \\
&\quad - L e_{12}^{-1} L e_{21}^{-1} k_n^4) k^2.
\end{aligned} \tag{20}$$

Assuming the principle of exchange of stabilities critical behavior is obtained by setting  $\sigma = 0$  [37], we have

$$C(k_m, k, 0) = 0, \tag{21}$$

where

$$\begin{aligned}
C(k_m, k, 0) &= (L e_{22}^{-1} L e_{11}^{-1} - L e_{12}^{-1} L e_{21}^{-1}) k_n^6 k^{-2} \\
&\quad - R_c (L e_{22}^{-1} - L e_{12}^{-1}) \\
&\quad - R'_c (L e_{11}^{-1} - L e_{12}^{-1}) \\
&\quad - R_t (L e_{22}^{-1} L e_{11}^{-1} - L e_{12}^{-1} L e_{21}^{-1}).
\end{aligned} \tag{22}$$

Now we can define a effective Rayleigh number as

$$\begin{aligned}
R &= [R_c (L e_{22}^{-1} - L e_{12}^{-1}) + R'_c (L e_{11}^{-1} - L e_{12}^{-1}) \\
&\quad + R_t (L e_{22}^{-1} L e_{11}^{-1} - L e_{12}^{-1} L e_{21}^{-1})] \\
&\quad \times (L e_{22}^{-1} L e_{11}^{-1} - L e_{12}^{-1} L e_{21}^{-1})^{-1}.
\end{aligned} \tag{23}$$

Hence, we can write  $R = k_n^6 k^{-2}$ . The lowest value is for  $n = 1$  and the instability starts at

$$\frac{\partial R}{\partial k^2} = 0. \quad (24)$$

This gives us  $k^2 = \pi^2/2$ , and the corresponding value of  $R$  will be given by  $R = 657.5 \sim 10^3$ .

This is when the instability will start. This convection in ternary mixtures can be used to analyse interesting geological phenomenon. This is because soils can exhibit semi-permeability and osmosis through these semipermeable soils can be driven by gradient in salt concentration [39, 40, 41]. In fact, an excess pressure has been observed to exist during the diffusion of salty water through certain soils [42, 43]. Thus, it will be interesting to analyse this phenomenon using stability analysis for ternary systems. In the limit when there is no cross diffusion,  $Le_{12}^{-1} = Le_{21}^{-1} = 0$ , we have the expected result  $R = R_c L_{11} + R'_c L_{22} + R_t$ . Thus, in this limit, the temperature Rayleigh number and all the concentration Rayleigh multiplied by there respective Lewis numbers add up to give the effective Rayleigh number. So, the effective Rayleigh number is enhanced by the existence of a salt gradient. This, also implies that convection can start much earlier in salty water than pure water. Thus, the evaporation of salty water at the surface of soils can set up interesting instabilities, governed by the ternary characteristic equation.

### 3 Binary Mixture

In this section we will neglect the effect due to impurities. However, we are not able to directly set  $Le_{22}^{-1} = Le_{21}^{-1} = Le_{12}^{-1} = 0$ , in the effective Rayleigh number. This is because the Green's function used to derive this effective Rayleigh number is not well defined for these values. This is because the differential operator  $D_0$  has a zero eigenvalue for these values of the Lewis numbers. So, its inverse does not exist. Thus, we can not set  $Le_{22}^{-1} = Le_{21}^{-1} = Le_{12}^{-1} = 0$ , in the effective Rayleigh number obtained by using this Green's function. To obtain a correct result we will have to repeat the above analysis for the binary solutions. However, it may be noted that all the analysis of the previous section, except the derivation of the Green's function remains, remains well defined, if we set  $Le_{22}^{-1} = Le_{21}^{-1} = Le_{12}^{-1} = 0$ . So, now we set,  $Le_{11} = Le$ ,  $Le_{22}^{-1} = Le_{21}^{-1} = Le_{12}^{-1} = 0$ ,  $\rho = \rho_0[1 + B_c C + B_t T]$ , and use the following equations

$$\begin{aligned} Pr^{-1} Dv_i &= -\partial_i p + \partial^2 v_i + R_c C \lambda_i + R_t T \lambda_i, \\ DC &= Le^{-1} \partial^2 C, \\ DT &= \partial^2 T. \end{aligned} \quad (25)$$

If we repeat the above analysis, we will obtain the following perturbative equations

$$\begin{aligned} D^{Pr} \partial^2 \lambda^i u_i - R_c \tilde{\partial}^2 \theta_c - R_t \tilde{\partial}^2 \theta_t &= 0, \\ D_{Le}^1 \theta_c - \lambda^i u_i &= 0, \\ D_1^1 \theta_t - \lambda^i u_i &= 0. \end{aligned} \quad (26)$$

Now we again define the following Green's functions

$$\begin{aligned} D_{Le}^1 G(r, r') &= \delta^3(r - r'), \\ D_1^1 G_t(r, r') &= \delta^3(r - r'). \end{aligned} \quad (27)$$

Thus, we can write

$$\begin{aligned} \theta_c(r) &= \int d^3 r' G(r, r') \lambda^i u_i(r'), \\ \theta_t(r) &= \int d^3 r' G_t(r, r') \lambda^i u_i(r'), \end{aligned} \quad (28)$$

because

$$\begin{aligned} D_{Le}^1 \theta_c(r) &= \int d^3 r' D_{Le}^1 G(r, r') \lambda^i u_i(r') \\ &= \int d^3 r' \delta(r - r') \lambda^i u_i(r') \\ &= \lambda^i u_i(r), \\ D_1^1 \theta_t(r) &= \int d^3 r' D_1^1 G_t(r, r') \lambda^i u_i(r') \\ &= \int d^3 r' \delta(r - r') \lambda^i u_i(r') \\ &= \lambda^i u_i(r). \end{aligned} \quad (29)$$

We can write

$$\begin{aligned} D^{Pr} \partial^2 \lambda^i u_i(r) &= R_c \tilde{\partial}^2 \int d^3 r' G(r, r') \lambda^i u_i(r') \\ &\quad + R_t \tilde{\partial}^2 \int d^3 r' G_t(r, r') \lambda^i u_i(r'). \end{aligned} \quad (30)$$

Now acting on this equation by  $D_1^1$  and  $D_{Le}^1$ , we get

$$D_1^1 D_{Le}^1 D^{Pr} \partial^2 \lambda^i u_i = R_c \tilde{\partial}^2 D_1^1 \lambda^i u_i + R_t \tilde{\partial}^2 D_{Le}^1 \lambda^i u_i(r'). \quad (31)$$

So, again using the boundary conditions that the even derivatives of  $\lambda^i \partial_i$  vanishes on  $\lambda^i u_i$ , we write the solution as [37]

$$\lambda^i u_i = A \sin n\pi \exp[i(k_x x + k_y y) + \sigma t], \quad (32)$$

Thus, we get

$$C(k_n, k, \sigma) = 0, \quad (33)$$

where

$$\begin{aligned} C(k_n, k, \sigma) &= (\sigma + k_n^2 L e^{-1})(P r^{-1} \sigma + k_n^2)(\sigma + k_n^2) k_n^2 \\ &\quad - (\sigma + k_n^2 L e^{-1}) k^2 R_t - (\sigma + k_n^2) k^2 R_c. \end{aligned} \quad (34)$$

So, in the case of temperature dependence critical behavior is obtained by setting  $\sigma = 0$ , we have [37]

$$C(k_m, k, 0) = 0, \quad (35)$$



where

$$C(k_n, k, 0) = k_n^6 k^{-2} Le^{-1} - Le^{-1} R_t - R_c. \quad (36)$$

Now here the effective Rayleigh number is

$$R = R_t + Le R_c \quad (37)$$

Hence, we can write  $R = k_n^6 k^{-2}$ . The lowest value is for  $n = 1$  and the instability starts at

$$\frac{\partial R}{\partial k^2} = 0. \quad (38)$$

This gives us  $k^2 = \pi^2/2$ , and the corresponding value of  $R$  will be again be given by  $R = 657.5 \sim 10^3$ .

Now the actual value for a binary mixture of bentonite and water are of the order,  $\rho_0 \sim 10^3$ ,  $\mu \sim 10^{-3}$ ,  $\kappa \sim 10^{-7}$ . Also the depth of the liquid  $l \sim 10^{-2}$  [22] and  $B_c \sim 1$ ,  $B_t \sim 10^{-1}$  [44], so we have  $R_t \sim 10^3$ , and  $R_c \sim 10^6$ . However, in the stability analysis the product of  $R_c$  and  $Le$  contributes to the occurrence of the instability and  $Le \sim 10^2$ . So, the contribution from the concentration gradient  $R_c Le \sim 10^8$  is much more than temperature gradient. The Marangoni effect is also measured by the Marangoni number  $M \sim 10^3$  [45]. Hence, the concentration gradient is the most dominant factor for convection to occur. The calculated value of the effective Rayleigh number due to the concentration gradient is much larger than the theoretical limit for the convection to occur. Hence, we hypothesize that it is responsible for the instability in the drying of a binary mixture of bentonite and water.

It has been observed that in a binary mixture of bentonite and water initially the convection takes place on the surface, but, eventually a layer forms on the surface and this inhibited further convection from taking place on the surface [22]. However, the convection continues in the lower depths of the soil. This observation rules out the possibility that this convection is due to the Marangoni effect. This is because in that case it would be surface driven phenomenon and would not continue at depth after halting at the surface. It will be interesting to explore the possibility that this behavior is due to the fact that in a soil the diffusivity and viscosity are strong functions of particle concentration. That is, the effective viscosity and diffusivity of a soil slurry increase dramatically with increasing particle concentration, and thus at sufficiently high particle concentrations the effective Rayleigh number will be reduced to below the critical value necessary for convection.

## 4 Conclusion

In this paper we have analysed the drying of a ternary mixture of soil particles and water mixed with impurities. We have demonstrated that the concentration gradient that forms during the process of evaporation of this mixture is the main driving force for the occurrence of a Rayleigh-Benard instability. This instability causes convection to take place, which in turn may have significant implications for the drying of soil crusts and the formation of patterned ground. In the ternary case the use of Green's functions allowed for a straight-forward calculation of the characteristic equation. For the binary case a separate calculation was required. This is because the differential operator that was inverted

using the Green's function in the ternary case, yields a zero eigenvalue in the binary case. Thus, it cannot be inverted and the ternary Green's function becomes ill defined for the binary case. As noted in the introduction, soil loss in agricultural areas depends critically on the formation of surface seals and crusts and the subsequent cracking of these crusts during drying of the ground. It is possible that the convection patterns give rise to weak points, which on drying gives rise to cracks [22]. Furthermore, patterns observed during the freezing and thawing of ground around the Arctic circle have also been explained as owing to convection of the soil. In this paper it was proposed that evaporation from the surface of thawing ground causes an unstable density profile to develop potentially leading to convection. The occurrence of similar patterns on Mars suggests that in the past the temperature on Mars may have been large enough for the ice to melt and evaporation to take place. It may be noted that it has been observed that drying on slanting slopes gives rise to rolls and drying on flat surface gives rise to hexagonal structures [32]. It will be interesting to perform stability analysis for these specific physical situations and see if the formation of these particular patterns can be explained.

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